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NON-CONSTANT CURVES OF GENUS 2 WITH INFINITE PRO-GALOIS COVERS

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ABSTRACT

For every odd prime number p, we give examples of non-constant smooth families of curves of genus 2 over fields of characteristic p which have pro-Galois (pro-étale) covers of infinite degree with geometrically connected fibers. The Jacobians of the curves are isomorphic to products of elliptic curves.

1. Introduction and Main Result

In the following, unless stated otherwise, a **curve** over a field is assumed to be smooth, proper and geometrically connected. This work is motivated by the following general problem.

Consider all curves of a fixed genus g over fields of a certain type (e.g. algebraically closed, local, finitely generated). Among these curves, does there exist a curve C over a field K which allows an infinite tower of non-trivial unramified

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covers $\cdots \to C_i \to \cdots \to C_0 = C$ such that for all i, C_i is also a (*geometrically connected*) curve over K and $C_i \to C$ is Galois?

We concentrate on this problem for *curves of genus* 2 *over finitely generated fields.* Some examples of curves of genus 2 over finite fields allowing such a tower of covers are known, and we address the question whether there also exist non-constant curves of genus 2 (over function fields) with such a tower. We show with explicit examples that this is the case.

Example 1.1: For every odd prime p, the (smooth, proper) curve of genus 2 given by the (affine) equation

$$Y^2 = (t-1)\cdot(1-t^{p-1})\cdot(X^2-1)\cdot(X^2-t^{p-1})\cdot(X^2-1-t-\cdots-t^{p-1})$$

over the global function field $K = \mathbb{F}_{p^2}(t)[\sqrt{t}, \sqrt{t-1}, \ldots, \sqrt{t-(p-1)}]$ allows such a tower of covers.

1.1. Further Background Information. We will use the following terminology. A **curve cover** over K is a surjective morphism of curves over K. A projective limit of a projective system of curves (where the morphisms are curve covers) over K is called a **pro-curve** over K. A **pro-curve cover** of a curve C over K is a surjective morphism $\pi : D \to C$ where D is a pro-curve.

Definition 1.2: A curve cover is **Galois** if it is unramified and the corresponding extension of function fields is Galois. A pro-curve cover is **pro-Galois** if it is a projective limit of a projective system of Galois covers.

For brevity, we speak of **pro-Galois curve covers** instead of pro-Galois pro-curve covers.

With this terminology, the general problem posed above can be reformulated as follows: *Are there curves of a fixed genus g over fields of certain types which allow pro-Galois curve covers of infinite degree?*

As an example consider the case where K is an algebraically closed field. Then a curve C over K allows a pro-Galois curve cover of infinite degree if and only if the genus of C is ≥ 1. The situation is very different over finitely generated fields.

No curve of genus 1 over a finitely generated field K has a pro-Galois curve cover of infinite degree. There even exists a universal bound $n = n(K)$ such that all Galois curve covers of a genus 1 curve with a rational point over K have degree $\leq n$. Indeed, by the Theorem of Mazur–Kamienny–Merel (see [16]) and an induction argument on the absolute transcendence degree of K, there exists a number $n = n(K)$ such that all elliptic curves over K have at most n K-rational torsion points; from this the assertion follows, see [7, Section 2] for details.

If K is finitely generated and C is a curve over K corresponding to the generic point of the moduli scheme of curves of a certain genus, the authors expect that C also does not have a pro-curve cover of infinite degree. But for special curves the situation changes. For every natural number $g \geq 3$ and every field K containing the 4-th roots of unity, explicit examples of curves over K of genus g having a pro-Galois curve cover of infinite degree can be given. If K is not finite over its prime field the curves can be chosen to be non-constant and even non-isotrivial (i.e., for every extension of the ground field, they stay non-constant); see [7, 6].

For finite fields, examples of curves of genus 2 with a pro-Galois curve cover of infinite degree are known (see [11, §3, Examples 2 and 3], [7]), but to the knowledge of the authors, no examples of non-constant curves of genus 2 with a pro-Galois curve cover have been known so far.

1.2. The Main Result. In this subsection we shall formulate the main result of this paper. We freely use some results of [1] which we recall at the beginning of Section 3. We begin with some notation.

Let p be an odd prime and N be an odd natural number. Let S be a smooth variety over a finite field of characteristic p. We assume that there is an S-isogeny

$$\tau : \mathcal{E} \to \mathcal{E}'$$

between two elliptic curves over S having degree N. The kernel of the multiplication by 2 of $\mathcal{E}$ is denoted by $\mathcal{E}[2]$, the image of the zero-section by $[0_{\mathcal{E}}]$. We denote by $\mathcal{E}[2]^{\#}$ the S-scheme $\mathcal{E}[2] - [0_{\mathcal{E}}]$. For the curve $\mathcal{E}'$ we use similar definitions.

Notation 1.3: Let $\mathcal{P} := \mathcal{E}/\langle -1\rangle, \mathcal{P}' := \mathcal{E}'/\langle -1\rangle$ with projections

$$\rho : \mathcal{E} \to \mathcal{P} \quad \text{and} \quad \rho' : \mathcal{E}' \to \mathcal{P}'.$$

Remark 1.4: The quotients $\mathcal{P}$ and $\mathcal{P}'$ are $\mathbb{P}^1$-bundles over S which are isomorphic to $\mathbb{P}^1_S$ if $\mathcal{E}[2]$ is S-isomorphic to the group scheme $(\mathbb{Z}/2\mathbb{Z})^2$.

There exists a unique S-isomorphism

$$\gamma : \mathcal{P} \tilde{\to} \mathcal{P}'$$

such that

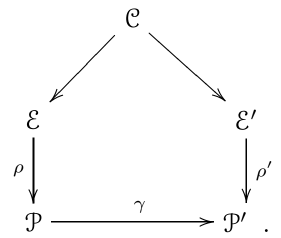
$$\rho' \circ \tau|_{\mathcal{E}[2]^{\#}} = \gamma \circ \rho|_{\mathcal{E}[2]^{\#}}.$$

Assumption 1.5: Let the following two equivalent conditions be satisfied.

- For no geometric point s of S, there exists an isomorphism $\alpha : \mathcal{E}_s \to \mathcal{E}'_s$ such that $\alpha|_{\mathcal{E}_s[2]} = \tau_s|_{\mathcal{E}_s[2]}$.
- $\gamma(\rho([0_{\mathcal{E}}]))$ and $\rho'([0_{\mathcal{E}'}])$ are disjoint.

Notation 1.6: Let $\mathcal{C}$ be the normalization of the integral scheme $\mathcal{E} \times_{\mathcal{P}'} \mathcal{E}'$ (where the product is with respect to $\gamma \circ \rho$ and ρ').

We thus have the commutative diagram



Now $\mathcal{C}$ is a curve of genus 2 over S and the morphisms $\mathcal{C} \to \mathcal{E}$, $\mathcal{C} \to \mathcal{E}$ are degree 2 covers. (For the terminology concerning relative curves, see Subsection 1.6.) It is the curve $\mathcal{C}$ which is the basic object of this article.

Our main result is the following theorem.

THEOREM 1: *We use the above notation and assume that Assumption 1.5 is satisfied. So $\mathcal{C}$ is a curve of genus 2 over S which is defined as in Notation 1.6.*

Then there exists a connected Galois cover $T \to S$ with Galois group, a (finite) elementary abelian 2-group such that the curve $\mathcal{C}_T$ over T has a pro-Galois curve cover whose Galois group $\mathcal{G}$ fits into an exact sequence

$$1 \to \prod_{i=1}^{\infty} A_{n_i} \to \mathcal{G} \to (\mathbb{Z}/2\mathbb{Z})^r \to 1$$

with $r \leq 4$, n_i are pairwise distinct natural numbers and A_{n_i} is the alternating group on n_i elements.

Remark 1.7: It is easy to see that the Jacobian variety $J_{\mathcal{C}}$ of $\mathcal{C}$ is isogenous to $\mathcal{E} \times \mathcal{E}'$. We shall prove the stronger result; that $J_{\mathcal{C}}$ is in fact isomorphic to $\mathcal{E} \times \mathcal{E}'$; see Proposition 3.5 as well as Subsection 5.1 for additional information.

1.3. Outline of the Proof. The proof of Theorem 1 is organized as follows:

In Section 2, we show some basic results on ramification loci on varying covers of a fixed elliptic curve by a fixed genus 2 curve over an algebraically closed field. In Section 3, we fix an isogeny $\tau : \mathcal{E} \to \mathcal{E}'$ as in Theorem 1 (but over a more general base scheme). Using results from [1], we first show that $\mathcal{C}$ is a curve of genus 2 whose Jacobian is isomorphic to $\mathcal{E} \times \mathcal{E}'$. We then show using the results of Section 2 that there are infinitely many covers from $\mathcal{C}$ to $\mathcal{E}$ with the same ramification and branch loci and pairwise distinct degrees. In Section 4, we show how our main result follows from the existence of infinitely many minimal genus 2 covers of a given elliptic curve with the same branch loci (under some conditions).

Theorem 1 follows by combining the final result of Section 2, Proposition 3.10, with the main result of Section 4, Proposition 4.1.

1.4. Derivation of Examples. Theorem 1 can be used to obtain Example 1.1. Let p be an odd prime, let $S := \mathbb{A}^1_{\mathbb{F}_p} - \{0, 1, \ldots, p-1\}$ and let us denote the coordinate on $\mathbb{A}^1_{\mathbb{F}_p}$ by t. Let $\mathcal{E}$ be the genus 1 curve over S given by

$$Y^2 Z - X(X - Z)(X - tZ) = 0.$$

We want to fix a section of $\mathcal{E}$ over S in order to turn $\mathcal{E}$ into an elliptic curve. For this, let $\mathcal{E}_a$ be the S-scheme given by $Y^2 - X(X-1)(X-t) = 0$ and let $a : S \to \mathcal{E}_a$ be section given by $X \mapsto t, Y \mapsto 0$. We have a natural inclusion $\iota : \mathcal{E}_a \hookrightarrow \mathcal{E}$ which is compatible with the projection to S, and $\iota \circ a$ is a section of $\mathcal{E}$ over S. We take this section as the zero-section.

Similarly, let $\mathcal{E}'$ be the elliptic curve over S given by

$$Y^2 Z - X(X - Z)(X - t^p Z) = 0$$

with zero-section determined by $X \mapsto t^p, Y \mapsto 0$, and let $\tau : \mathcal{E} \to \mathcal{E}'$ be the Frobenius endomorphism given by

$$X \mapsto X^p,\ Y \mapsto Y^p, Z \mapsto Z^p.$$

We note that the quotients $\mathcal{E} \to \mathcal{P}, \mathcal{E}' \to \mathcal{P}'$ can be identified with the usual projections to $\mathbb{P}^1_S$ (to the "X-coordinate"), and under these identifications, $\gamma : \mathcal{P} \to \mathcal{P}'$, becomes the identity. We also note that we have chosen S in such a way that the assumptions of Theorem 1 are satisfied. The curve $\mathcal{C}$ defined in Notation 1.6 is the normalization of $\mathcal{E} \times_{\mathbb{P}^1_S} \mathcal{E}'$. The field K in the introduction is obtained as the function field of the "maximal" connected Galois cover $T \to S$ which has an elementary abelian 2-group as Galois group, that is, it is the maximal Galois extension of $\mathbb{F}_p(t)$ which is unramified outside the places corresponding to $0, 1, \ldots, p-1$ and the place ∞ and has an elementary abelian 2-group as Galois group. (The explicit description of the field can, for example, be obtained via Kummer theory.) The curve in the introduction is the restriction of the curve $\mathcal{C}_T$ to the generic fiber of T, that is, to $\mathrm{Spec}(K)$.

Further examples can be obtained as follows:

Let p be an odd prime and $N \geq 3$ and assume that p and N are coprime. It is well-known that there exists a fine moduli scheme for elliptic curves with cyclic subgroups of order N and level-4-structure (of fixed determinant ζ_4) over schemes over $\mathbb{F}_p(\zeta_4)$. (p is odd.) Let us denote this scheme by $Y_0(N;4)_{\mathbb{F}_p(\zeta_4)}$: it is a smooth affine curve over $\mathbb{F}_p(\zeta_4)$.

Let $\tau : \mathcal{E} \to \mathcal{E}'$ be the universal isogeny over $Y_0(N;4)_{\mathbb{F}_p(\zeta_4)}$. There exists a uniquely determined open subscheme S of $Y_0(N;4)_{\mathbb{F}_p(\zeta_4)}$ such that the assumptions of Theorem 1 are satisfied. Obviously, S is non-empty (as otherwise the universal elliptic curve over $X_0(N;4)_{\mathbb{F}_p(\zeta_4)}$ would have complex multiplication).

If we now apply Definition 1.6 to the restriction of the universal isogeny $\tau : \mathcal{E} \to \mathcal{E}'$ to S, we obtain a curve $\mathcal{C}$ over S. We can now apply Theorem 1 to this curve.

1.5. Open Problems.

Problem 1.8: It is an open problem whether there exists a curve of genus 2 over a finitely generated field of characteristic 0 with a pro-Galois curve cover of infinite degree. The existence of such a curve would of course be implied by the existence of a genus 2 curve $\mathcal{C}$ with a pro-Galois curve cover of infinite degree over an open part of $\mathrm{Spec}(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the principal order in a number field. In Subsection 5.2 we discuss difficulties occurring when one tries to adapt the proof of Theorem 1 to the "mixed-characteristic case".

There is a sharpening of the condition on a curve over a field having a pro-Galois curve cover of infinite degree.

Let C be a curve over a field K and let $P \in C(K)$. Following the ideas of [11], the K-rational geometric fundamental group is introduced in [7]. Let us denote this group by $\pi_1^{\text{geo}}(C, P)$. The group $\pi_1^{\text{geo}}(C, P)$ classifies pro-étale (curve) covers $c : D \to C$ such that P splits completely. If $\deg(c)$ is finite, this means that the topological point P has exactly $\deg(c)$ preimages under c. Thus the group $\pi_1^{\text{geo}}(C, P)$ is infinite if and only if a pro-étale (or pro-Galois) (curve) cover of $D \to C$ of infinite degree exists such that P splits completely.

For every prime p, there exists a curve over a finite field K of characteristic p with infinite K-rational geometric fundamental group. There also exists a non-isotrivial curve over a function field in one variable over a finite field of given prime characteristic p with infinite K-rational fundamental group; this follows from [6, Theorem 4.3].

Moreover, for every natural number $g \geq 3$ and every field K containing the 4-th roots of unity of characteristic congruent 3 modulo 4, explicit examples of curves over K of genus g having an infinite K-rational fundamental group can be given. If K is not finite over its prime field the curves can be chosen to be non-isotrivial; see [6, Theorem 1.1].

Problem 1.9: It is an open problem whether any curve over a finitely generated field K of characteristic 0 with infinite K-rational geometric fundamental group exists,[1] and the case $g = 2$ remains open in any characteristic.

1.6. Terminology and Facts. The usage of "Galois" follows [8, Exposé V]. In particular, if Y is a locally noetherian, integral, normal scheme, X is an integral scheme and $f : X \to Y$ is a finite surjective morphism, then f is a Galois cover if and only if it is unramified and the corresponding extension of function fields is Galois; see [8, Exposé I, Corollaire 9.11] and [8, Exposé V, §8].

Let S be a connected locally noetherian scheme. A **(relative) curve** of genus g over S is a smooth, proper S-scheme all of whose geometric fibers are connected non-singular curves. We denote curves over S by $\mathcal{C}, \mathcal{D}, \mathcal{E}, \ldots$. If S is the spectrum of a field, we also use the notation $C, D, E, \ldots$.

[1] The proof of [7, Theorem 4.22] and thus also the proof of the "Result" at the end of the introduction of [7] are incorrect: As stated in [6, Remark 5.4], the condition $\text{char}(K) \equiv 3 \bmod 4$ has to be inserted.

By the first corollary to the Theorem in [20, Chapter II, §5], the Euler-Poincaré characteristic of the geometric fibers of a relative curve is constant, thus the genus of the fibers is also constant; we define the **genus** of a relative curve to be the genus of any of its fibers.

We will often use the fact that a curve over a regular scheme is regular; see [8, Exposé II, Proposition 3.1].

A **curve cover** of a curve $\mathcal{C}$ over S is a finite and flat S-morphism $\pi : \mathcal{D} \to \mathcal{C}$, where $\mathcal{D}$ is also a curve over S.

Note that an S-morphism $\pi : \mathcal{D} \to \mathcal{C}$ between two curves over S which induces curve covers on the fibers is automatically finite and flat, that is, it is a curve cover. (The morphism is fiberwise flat ([10, Proposition 9.7]) and thus flat by [9, IV (11.3.11)], it is finite by [10, Corollary 4.8] and [9, IV (8.11.1)].)

A **pro-curve** over S is a projective limit of a projective system of curves with respect to morphisms which are curve covers over S.

A **pro-curve cover** of a curve $\mathcal{C}$ over S is a projective limit of a projective system of curve covers of $\mathcal{C}$. A pro-Galois pro-curve cover is abbreviated as **pro-Galois curve cover**, and its automorphism group is called **Galois group**.

We shall identify effective Cartier divisors on a locally noetherian integral scheme X with the locally principal closed subschemes of X (cf., [10, Remark 6.17.1], [9, 21.2.12]). (This means in particular that Cartier divisors on an integral, regular schemes are identified with closed subschemes of codimension 1.) If additionally X is an S-scheme, then under this identification, relative effective Cartier divisors on X over S correspond to locally principal closed subschemes of X which are flat over S; see [18, Definition 3.4].

Let S be a scheme. If S is integral, we denote its function field by $\kappa(S)$. If X is an S-scheme and $T \to S$ is a morphism, we denote the pull-back of X to T by X_T. If $\alpha : X \to Y$ is a morphism of S-schemes, we denote its pull-back via T by $\alpha_T : X_T \to Y_T$. Note that if X is a curve over S and T is connected and locally noetherian, X_T is a curve over T.

If $\mathcal{A}$ is an abelian scheme over S, the dual abelian scheme $\widehat{\mathcal{A}}$ exists by [3, Theorem 1.9]. If $\varphi : \mathcal{A} \to \mathcal{B}$ is a homomorphism of abelian schemes over S, we denote the dual homomorphism by $\widehat{\varphi} : \widehat{\mathcal{B}} \to \widehat{\mathcal{A}}$.

If $\mathcal{A}$ and $\mathcal{B}$ are two abelian schemes over S, we denote the product over S by $\mathcal{A} \times \mathcal{B}$. If $\mathcal{A}_1, \ldots, \mathcal{A}_n$ and $\mathcal{B}_1, \ldots, \mathcal{B}_m$ are abelian schemes over S, we have a canonical isomorphism between the group of homomorphisms from

$\prod_j \mathcal{A}_j$ to $\prod_i \mathcal{B}_i$ and the group of **matrices** $(a_{i,j})_{i=1\ldots,m,j=1,\ldots,n}$, where $a_{i,j} \in \mathrm{Hom}_S(\mathcal{A}_j, \mathcal{B}_i)$. The composition of homomorphisms (if applicable) corresponds to the multiplication of the matrices. If we use matrices, we identify $\mathbb{Z}$ with its image in any endomorphism ring.

If $\mathcal{C}$ is a curve over S, we denote its Jacobian by $J_{\mathcal{C}}$. The dual of the Jacobian is denoted by $J_{\mathcal{C}}$ and the canonical principal polarization on $J_{\mathcal{C}}$ by $\lambda_{\mathcal{C}} : J_{\mathcal{C}} \to \widehat{J_{\mathcal{C}}}$; (cf., [19, Proposition 6.9]). If $\pi : \mathcal{D} \to \mathcal{C}$ is a curve cover over S, we have canonical homomorphisms $\pi^* : J_{\mathcal{C}} \to J_{\mathcal{D}}$ and $\pi_* := \lambda_{\mathcal{C}}^{-1}\, \widehat{\pi^*}\, \lambda_{\mathcal{D}} : J_{\mathcal{D}} \to J_{\mathcal{C}}$.

Let $\mathcal{E}$ be an elliptic curve over S, $\mathcal{C}$ a genus 2 curve over S, $\pi : \mathcal{C} \to \mathcal{E}$ a curve cover. We call π **minimal** if it does not factor through a non-trivial isogeny $\mathcal{E}_1 \to \mathcal{E}$. Note that this is equivalent to $\ker(\pi_* : J_{\mathcal{C}} \to \mathcal{E})$ being an elliptic curve, and it is equivalent to $\pi^* : \mathcal{E} \to J_{\mathcal{C}}$ being a closed immersion, see the beginning of Section 2 of [12] and Point 7) in Section 7 of [12].

Let $\pi : X \to Y$ be a finite morphism of locally noetherian schemes. Then the image $\pi(x) \in Y$ of a ramification point $x \in X$ is called **branch point**. The set of ramification points as well as the set of branch points are closed in X and Y respectively (the set of ramification points is closed because it the support of $\Omega_{X/Y}$, and the set of branch points is then the image of a closed set under a finite morphism, thus also closed). The corresponding schemes with the reduced induced scheme structures are called **ramification locus** and **branch locus**, respectively.

We say that a field extension $K|k$ is **regular** if $K \otimes_k \overline{k}$ is a domain ($\overline{k}$ = algebraic closure of k) (cf., [14, VIII, §4]). This notion should not be confused with the notion of a regular scheme.

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2. The Ramification Loci of Covers of Genus 2 of an Elliptic Curve

Let $\overline{\kappa}$ be an algebraically closed field. When we speak of the **intersection** of two curves on a smooth surface, we mean the scheme-theoretic intersection. We will use some easy results from intersection theory which we recall in an appendix to this section.

Let C be a curve of genus 2 over $\overline{\kappa}$.

PROPOSITION 2.1: *Let P be a closed point of C, let $c : C \to E$ be a cover which maps P to 0_E, let $\iota : C \to J_C$ be the canonical immersion which maps P to 0_E. Then the following assertions are equivalent.*

- *c is unramified in P.*
- *$\iota(C)$ and $\ker(c_*)$ intersect transversely in $\iota(P)$ (inside J_C).*

Proof. Let $c^{-1}(0_E) = C_{0_E} = C \times_E 0_E$ be the fiber of c at 0_E. Note that $c = c_* \circ \iota$. Consider the following Cartesian diagram

$$\begin{array}{ccc} c^{-1}(0_E) = C \times_E 0_E & \hookrightarrow & C \\ \downarrow & & \downarrow \iota \\ \ker(c_*) & \hookrightarrow & J_C \\ \downarrow & & \downarrow c_* \\ 0_E & \hookrightarrow & E \end{array}$$

We know that $c^{-1}(0_E)$ is a 0-dimensional closed subscheme on C, and its support contains P. Both statements of the proposition are equivalent to the multiplicity of $c^{-1}(0_E)$ at P being 1. ∎

PROPOSITION 2.2: *Let $c_1 : C \to E_1$, $c_2 : C \to E_2$ be minimal covers such that there does not exist an isomorphism of curves $\sigma : E_1 \tilde{\to} E_2$ with $\sigma \circ c_1 = c_2$. Then $\ker(c_{1*})$ and $\ker(c_{2*})$ are abelian subvarieties of J_C which intersect in a finite group scheme. There is the following alternative:*

- Either *the intersection of* $\ker(c_{1*})$ *and* $\ker(c_{2*})$ *is a reduced (étale) finite group scheme and the ramification loci of c_1 and c_2 are disjoint.*
- or *the intersection of* $\ker(c_{1*})$ *and* $\ker(c_{2*})$ *is a non-reduced finite group scheme and the ramification loci of c_1 and c_2 are equal.*

Note that if $\mathrm{char}(\overline{\kappa}) = 0$ *the first alternative holds.*

Proof. The fact that the abelian subvarieties $\ker(c_{1*})$ and $\ker(c_{2*})$ of J_C intersect in a finite group scheme follows because by assumption there does not exist a $\tilde{\sigma} \in \mathrm{Iso}(E_1, E_2)$ with $\tilde{\sigma} \circ c_{1*} = c_{2*}$.

Note that $\ker(c_{1*})$ and $\ker(c_{2*})$ intersect transversely at 0 if and only if the intersection of the two elliptic curves is a reduced (étale) finite group scheme; see also Lemma 2.6).

Let P be a closed point of C. By composing the c_i with suitable translations on E_i, we can assume that $c_i(P) = 0_{E_i}$ $(i = 1, 2)$. We can now apply the previous proposition. By the fact that "non-transversal intersection at P" is an equivalence relation (see Lemma 2.7), we conclude:

If $\ker(c_{1*}) \cap \ker(c_{2*})$ is a non-reduced finite group scheme, P is a ramification point of c_1 if and only if it is a ramification point of c_2.

If, on the other hand, $\ker(c_{1*}) \cap \ker(c_{2*})$ is a reduced finite group scheme, P cannot be a ramification point of both c_1 and c_2. ∎

Proposition 2.3: *Assume that* $\mathrm{char}(\overline{\kappa}) = p > 0$. *Let* $c_1 : C \to E$ *and* $c_2 : C \to E$ *be minimal covers. Let* $\overline{c}_{1*}, \overline{c}_{2*} \in \mathrm{Hom}_{\overline{\kappa}}(J_C, E) \otimes_{\mathbb{Z}} \mathbb{F}_p$ *be the induced elements, and assume that* $\overline{c}_{1*} = \overline{c}_{2*}$. *Then the ramification loci of* c_1 *and* c_2 *are equal.*

Proof. Under the assumption of the proposition, $\ker(c_{1*})[p]$ is equal to $\ker(c_{2*})[p]$ (as closed subschemes of J_C). In particular, $\ker(c_{1*}) = \ker(c_{2*})$ or $\ker(c_{1*}) \cap \ker(c_{2*})$ is non-reduced. The result follows with the last proposition. ∎

Appendix to Section 2: Facts about Intersection Theory. Let $\overline{\kappa}$ be an algebraically closed field, let X be a smooth surface over $\overline{\kappa}$, and let $P \in X$ be a closed point. Let Y be a curve in X. As always we assume that Y is smooth. For the following arguments it would suffice that X and Y are smooth at P. Note that X is locally factorial, so if $Y \hookrightarrow X$ is a 1-dimensional closed subscheme of X containing P with canonical closed immersion $\iota_Y : Y \hookrightarrow X$, the kernel of $\iota_Y^{\#} : \mathcal{O}_{X,P} \to \mathcal{O}_{Y,P}$ is generated by a single element $f \in \mathcal{O}_{X,P}$ (unique up to a unit). We call such an f a **local equation** of Y at P. Note that $f \in \mathfrak{m}_{X,P}$.

Definition 2.4: Let Y_1 and Y_2 be two curves on X such that $P \in Y_1, P \in Y_2$. Then Y_1 and Y_2 **intersect transversely at** P if the local equations of Y_1 and Y_2 at P generate $\mathfrak{m}_{X,P}$.

Lemma 2.5: *Let* Y *be a curve in* X, *let* P *be a closed point on* Y. *Then the local equation of* Y *at* P *does not lie in* $\mathfrak{m}_{X,P}^2$.

Proof. This follows easily from the fact that X is regular and 2-dimensional whereas Y is regular and 1-dimensional. ∎

For the following lemma, note that the surjection $\mathfrak{m}_{X,P} \to \mathfrak{m}_{Y_i,P}$ induces by dualization injections of (Zariski) tangent spaces $T_{Y_i,P} \hookrightarrow T_{X,P}$.

LEMMA 2.6: *Let $Y_i \subset X$ $(i = 1, 2)$ be two curves in X meeting in P. Let $Y_1 \cap Y_2 := Y_1 \times_X Y_2$ be the scheme-theoretic intersection of Y_1 and Y_2. The following statements are equivalent:*

a) *Y_1 and Y_2 intersect transversely at P.*
b) *Viewed as elements in $\mathfrak{m}_{X,P}/\mathfrak{m}^2_{X,P}$, the local equations of Y_1 and Y_2 at P are linearly independent.*
c) *The canonical homomorphism $\mathfrak{m}_{X,P}/\mathfrak{m}^2_{X,P} \to \mathfrak{m}_{Y_1,P}/\mathfrak{m}^2_{Y_1,P} \times \mathfrak{m}_{Y_2,P}/\mathfrak{m}^2_{Y_2,P}$ is an isomorphism.*
d) *The canonical homomorphism $T_{Y_1,P} \times T_{Y_2,P} \to T_{X,P}$ is an isomorphism.*
e) *There exists a neighbourhood U of P such that $(Y_1 \cap Y_2)|_U$ is equal to the closed subscheme given by the closed immersion* Spec $(\overline{\kappa}) \hookrightarrow U$ *at P.*

Proof. a) and b) are equivalent by Nakayama's Lemma. The equivalence of b) and c) is easy, and d) is a "dual formulation" of c). The local ring of $Y_1 \cap Y_2$ at P is isomorphic to $\mathcal{O}_{X,P}/(f_1, f_2)$. This implies that a) holds if and only if the local ring of $Y_1 \cap Y_2$ at P is isomorphic to Spec$(\overline{\kappa})$. This in turn is equivalent to e). ∎

We define a relation on the set of all curves Y lying in X and meeting P as follows: Y_1 is equivalent to Y_2 if and only if Y_1 and Y_2 do not intersect transversely at P.

LEMMA 2.7: *The relation just defined is an equivalence relation.*

Proof. We only have to prove the transitivity. This follows from point b) in the above equivalences with Lemma 2.5. ∎

3. Minimal Covers with Prescribed Branch Loci

In this section, we first show how the results of [1] imply the statements which were implicitly used in Subsection 1.2. Then we use the results of the previous section to derive a preliminary result on our way to prove Theorem 1.

Let us start off as in Subsection 1.2: Let p be an odd prime number, and let N be an odd natural number. Let S be a locally noetherian, integral, regular scheme of characteristic p. Let $\tau : \mathcal{E} \to \mathcal{E}'$ be an isogeny of elliptic curves over S of degree N.

Also, as in Subsection 1.2, let $\mathcal{E}[2]^{\#} := \mathcal{E}[2] - [0_{\mathcal{E}}]$, where $[0_{\mathcal{E}}]$ is the Cartier divisor associated to the zero-section $0_{\mathcal{E}}$ of $\mathcal{E}$ (similar definitions for $\mathcal{E}'$), let $\mathcal{P} := E/\langle -1\rangle, \mathcal{P}' := E'/\langle -1\rangle$ with projections $\rho : \mathcal{E} \to \mathcal{P}$ and $\rho' : \mathcal{E}' \to \mathcal{P}'$.

Let $\psi := \tau|_{\mathcal{E}[2]} : \mathcal{E}[2] \to \mathcal{E}'[2]$.

It is well-known that $\mathcal{P}$ is isomorphic to the $\mathbb{P}^1$-bundle $\mathbb{P}(q_*(\mathcal{L}(0_{\mathcal{E}})))$, where $q : \mathcal{E} \to S$ is the structure morphism (see for example the discussion preceding Theorem 2 in [1]), and a similar statement holds for $\mathcal{P}'$. Note that $\mathcal{E}[2]^{\#} \to S$ and $\mathcal{E}'[2]^{\#} \to S$ are étale covers of degree 3 and $\mathcal{E}[2]^{\#} \to \mathcal{P}$ as well as $\mathcal{E}'[2]^{\#} \to \mathcal{P}'$ are closed immersions. By [1, Propsition B.4], there exists a unique S-isomorphism $\gamma : \mathcal{P} \to \mathcal{P}'$ with $\gamma \circ \rho|_{\mathcal{E}[2]^{\#}} = \rho' \circ \psi$. (The fact that $\mathcal{P}$ is isomorphic to $\mathbb{P}^1_S$ if $\mathcal{E}[2] \simeq (\mathbb{Z}/2\mathbb{Z})^2$ also follows from [1, Propsition B.4].)

Lemma 3.1: *The conditions*

- *for no geometric point s of S, there exists an isomorphism $\alpha : \mathcal{E}_s \to \mathcal{E}'_s$ such that $\alpha|_{\mathcal{E}_s[2]^{\#}} = \psi$*
- *$\gamma \circ \rho([0_{\mathcal{E}}])$ and $\rho'([0_{\mathcal{E}'}])$ are disjoint*

are equivalent.

Proof. It is enough to prove the lemma in the case that S is the spectrum of an algebraically closed field field. In this situation one can use [1, Lemma A.1]. ∎

For the rest of this section, we assume that the conditions of Lemma 3.1 are satisfied.

Lemma 3.2: *$\mathcal{E} \times_{\mathcal{P}'} \mathcal{E}'$ is integral.*

As in Notation 1.3, let $\mathcal{C}$ be the normalization of $\mathcal{E} \times_{\mathcal{P}'} \mathcal{E}'$, and let

$$\pi : \mathcal{C} \to \mathcal{E}, \pi' : \mathcal{C} \to \mathcal{E}'$$

be the canonical projections.

Proposition 3.3: *$\mathcal{C}$ is a curve of genus 2 over S and the maps π, π' are degree 2 covers.*

For Lemma 3.2 and Proposition 3.3, see [1, Proposition 4.4].

In the notation of [1] ([1, Definition 2.7, Proposition 2.15]), $(\mathcal{C}, \pi, \pi')$ is a "normalized symmetric pair" corresponding to $(\mathcal{E}, \mathcal{E}', \psi)$. This means that

- $\ker(\pi_*) = \operatorname{Im}((\pi')^*)$ and $\ker(\pi^*) = \operatorname{Im}(\pi'_*)$,
- $\pi_* W_{\mathcal{C}} = \mathcal{E}[2]^\#$ and $\pi'_* W_{\mathcal{C}} = \mathcal{E}'[2]^\#$,
- $\pi^*|_{\mathcal{E}[2]} = (\pi')^* \circ \psi$.

Here, $W_{\mathcal{C}}$ is the Weierstraß divisor of $\mathcal{C}$ and $\mathcal{E}[2]^\# := \mathcal{E}[2] - [0_{\mathcal{E}}]$, where $[0_{\mathcal{E}}]$ is the Cartier divisor associated to the zero-section $0_{\mathcal{E}}$ of $\mathcal{E}$ (similar definition for $\mathcal{E}'[2]^\#$).

Remark 3.4: Even without the assumption that S is regular, one can prove that a genus 2 curve $\mathcal{C}$ over S and two degree 2 covers π, π' with the above conditions exist. Moreover, the triple $(\mathcal{C}, \pi, \pi')$ is, up to unique isomorphism (defined in an obvious way), uniquely determined by the three conditions. On the other hand, if the two equivalent conditions in Lemma 3.1 are not satisfied, no normalized symmetric pair corresponding to $(\mathcal{E}, \mathcal{E}, \psi)$ exists. For more information on these issues, we refer the reader to [1].

Let $\delta : \mathcal{E} \times \mathcal{E}' \to J_{\mathcal{C}}$ be the homomorphism given by the matrix $\begin{pmatrix} \pi^* & (\pi')^* \end{pmatrix}$. (We identify $\mathcal{E}$ and $\mathcal{E}'$ with their dual abelian varieties.)

By [1, Proposition 2.14], the kernel of δ is $\operatorname{Graph}(-\psi) = \operatorname{Graph}(\psi)$, and the pull-back of the canonical principal polarization of $J_{\mathcal{C}}$ via π is twice the canonical product polarization of $\mathcal{E} \times \mathcal{E}'$.

Proposition 3.5: *The Jacobian $J_{\mathcal{C}}$ is isomorphic to $\mathcal{E} \times \mathcal{E}'$.*

Proof. As said above, $J_{\mathcal{C}}$ is isomorphic to $(\mathcal{E} \times \mathcal{E}')/\operatorname{Graph}(-\psi)$. The latter is in turn isomorphic to $\mathcal{E} \times \mathcal{E}'$. In fact, the isogeny

$$\Phi : \mathcal{E} \times \mathcal{E}' \to \mathcal{E} \times \mathcal{E}'$$

given by the matrix $\left(\begin{smallmatrix} 2 & 0 \\ \tau & 1 \end{smallmatrix}\right)$ has kernel $\operatorname{Graph}(-\psi)$, thus it induces an isomorphism $(\mathcal{E} \times \mathcal{E}')/\operatorname{Graph}(-\psi) \tilde{\to} \mathcal{E} \times \mathcal{E}'$. ∎

We use the group structure on $\mathcal{E}$ and obtain for all $a, b \in \mathbb{Z}$ a morphism $a\pi + b\hat{\tau}\pi' : \mathcal{C} \to \mathcal{E}$. We have a bilinear form

$$\beta : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \operatorname{End}(\mathcal{E}),\ ((a,b),(c,d)) \mapsto (a\pi + b\hat{\tau}\pi')_* \, (c\pi + d\hat{\tau}\pi')^*$$

with $\beta((a,b),(a,b)) = \deg(a\pi + b\widehat{\tau}\pi')$. As by definition $\pi_* \circ (\pi')^* = 0$ and $\pi'_* \circ \pi^* = 0$, we have $\beta((a,b),(c,d)) = 2ac + 2Nbd \in \mathbb{Z}$. This implies that

$$\deg(a\pi + b\widehat{\tau}\pi') = 2a^2 + 2Nb^2. \tag{1}$$

Let us fix the following notation: For $(a,b) \in \mathbb{Z}^2$, we denote the ramification locus of $a\pi + b\widehat{\tau}\pi'$ by $V_{(a,b)}$ and the branch locus by $\Delta_{(a,b)}$. Further, we set $V := V_{(1,0)}$ and $\Delta := \Delta_{(1,0)}$. The following lemma is [1, Proposition 3.11].

Lemma 3.6: *$\pi'|_V$ is the zero-element in the abelian group $\mathcal{E}'(V)$.*

Proposition 3.7: *If $p|b$, then $V_{(1,b)} = V$ and $\Delta_{(1,b)} = \Delta$.*

Proof. Let $\overline{\kappa}$ be any algebraically closed field, and let s be any point of $S(\overline{k})$. By Proposition 2.3, π_s and $(\pi + b\widehat{\tau}\pi')_s : \mathcal{C}_s \to \mathcal{E}_s$ have the same ramification locus.

Now let x be a $\overline{\kappa}$-valued point of $\mathcal{C}$ lying over s. Then the support of x is a ramification point of π if and only if it is a ramification point of π_s: both conditions are equivalent to the inclusion of $\overline{\kappa}$ into the local ring of x in the geometric fiber $\mathcal{C}_{\pi(x)}$ being not surjective. A corresponding statement holds for $\pi + b\widehat{\tau}\pi'$.

This implies that π and $\pi + b\widehat{\tau}\pi' : \mathcal{C} \to \mathcal{E}$ have the same ramification locus, that is, the underlying sets of $V_{(1,b)}$ and V are equal. As both schemes are endowed with the reduced induced scheme structure, we have that $V_{(1,b)} = V$.

By Lemma 3.6, it follows that

$$(\pi + b\widehat{\tau}\pi')|_V = \pi|_V + b\widehat{\tau}\pi'|_V = \pi|_V,$$

In particular, V is mapped under $\pi + b\widehat{\tau}\pi'$ to Δ. This implies that $\Delta_{(1,b)} = \Delta$. ∎

Proposition 3.8: *If $2|b$, then $\pi + b\widehat{\tau}\pi'$ is minimal of degree $2 + 2Nb^2$.*

Proof. The statement on the degree follows from (1).

To show that $\pi + b\widehat{\tau}\pi'$ is minimal, that is, that it does not factor through a non-trivial isogeny $\widetilde{\mathcal{E}} \to \mathcal{E}$, it suffices to show that the homomorphism

$$\pi_* + b\widehat{\tau}\pi'_* : J_{\mathcal{C}} \to \mathcal{E}$$

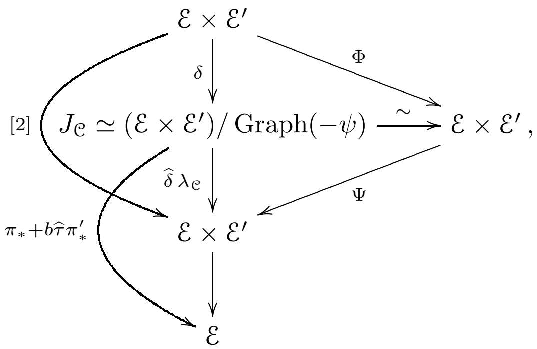
does not factor through a non-trivial isogeny $\widetilde{\mathcal{E}} \to \mathcal{E}$.

So assume that $\pi_* + b\widehat{\tau}$ factors through the isogeny $\widetilde{\mathcal{E}} \to \mathcal{E}$. Then

$$(\pi_* + b\widehat{\tau}\pi'_*) \circ \pi^* : \mathcal{E} \to \mathcal{E}$$

also factors through $\widetilde{\mathcal{E}} \to \mathcal{E}$. Now $(\pi_* + b\widehat{\tau}\pi'_*) \circ \pi^* = 2 \cdot \mathrm{id}_{\mathcal{E}}$. This implies that the degree of the isogeny $\tilde{\mathcal{E}} \to \mathcal{E}$ divides 4.

To rule out that the degree is 2 or 4, we consider the commutative diagram



where Φ is given as in Proposition 3.5, $\Psi : \mathcal{E} \times \mathcal{E}' \to \mathcal{E} \times \mathcal{E}'$ is given by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ -\tau & 2 \end{smallmatrix}\right)$, and the last vertical arrow is given by the matrix $\left(\begin{smallmatrix} 1 & b\widehat{\tau} \end{smallmatrix}\right)$. The homomorphism

$$\widehat{\delta}\,\lambda_{\mathcal{C}} : J_{\mathcal{C}} \to \mathcal{E} \times \mathcal{E}'$$

is given by the matrix $\left(\begin{smallmatrix} \pi_* \\ \pi'_* \end{smallmatrix}\right)$.

Under the horizontal isomorphism in the diagram, $\pi_* + b\widehat{\tau}\pi'_*$ is given by the matrix $(1 - bN \quad 2b\widehat{\tau})$. Let $\iota : \mathcal{E} \to \mathcal{E} \times \mathcal{E}' \simeq J_{\mathcal{C}}$ be the inclusion of the first summand. Then $(\pi_* + b\widehat{\tau}\pi'_*) \circ \iota = (1 - bN) \cdot \mathrm{id}_{\mathcal{E}}$. As by assumption $1 - bN$ is odd, the degree of $\widetilde{\mathcal{E}} \to \mathcal{E}$ cannot be divisible by 2. It is thus an isomorphism.

We conclude that $\pi + b\widehat{\tau}\pi' : \mathcal{C} \to \mathcal{E}$ is minimal. ∎

The last two propositions imply:

Proposition 3.9: *The covers $\pi_i := \pi + 2ip\widehat{\tau}\pi' : \mathcal{C} \to \mathcal{E}$ ($i \in \mathbb{N}$) are minimal of degree $2 + 8N(ip)^2$ and have the same ramification loci and the same branch loci.*

We summarize the results stated in Lemma 3.1, Lemma 3.2 and in Propositions 3.3, 3.5 and 3.9 in the following proposition.

Proposition 3.10: *Let p be an odd prime, let N be odd. Let S be a locally noetherian, integral, regular scheme of characteristic p, let $\tau : \mathcal{E} \to \mathcal{E}'$ be a cyclic isogeny of degree N over S.*

Then there exists a unique S-isomorphism $\gamma : \mathcal{P} \tilde{\to} \mathcal{P}'$ such that $\rho' \circ \tau|_{\mathcal{E}[2]^\#} = \gamma \circ \rho|_{\mathcal{E}[2]^\#}$. Assume that the following two equivalent conditions are satisfied.

- *For no geometric point s of S, there exists an isomorphism $\alpha : \mathcal{E}_s \to \mathcal{E}'_s$ such that $\alpha|_{\mathcal{E}_s[2]} = \tau_s|_{\mathcal{E}_s[2]}$.*
- *$\gamma(\rho([0_\mathcal{E}]))$ and $\rho'([0_{\mathcal{E}'}])$ are disjoint.*

Let $\mathcal{C}$ be the normalization of the integral scheme $\mathcal{E} \times_{\mathcal{P}'} \mathcal{E}'$ (where the product is with respect to $\gamma \circ \rho$ and ρ'). Then $\mathcal{C}$ is a curve of genus 2 with $J_\mathcal{C} \simeq \mathcal{E} \times \mathcal{E}'$, the canonical morphisms $\pi : \mathcal{C} \to \mathcal{E}$ and $\pi' : \mathcal{C} \to \mathcal{E}'$ are degree 2 covers and there exists a sequence $(\pi_i)_{i \in \mathbb{N}_0}$ of minimal covers $\pi_i : \mathcal{C} \to \mathcal{E}$ with pairwise distinct degrees and $\pi_0 = \pi$ such that the ramification loci as well as the branch loci of the π_i are all equal.

4. Pro-Galois Curve Covers of Infinite Degree

The goal of this section is to prove the following proposition which together with Proposition 3.10 implies Theorem 1.

PROPOSITION 4.1: *Let S be an integral, regular scheme of finite type over $\mathbb{Z}[1/2]$ or more generally a locally noetherian, integral, regular scheme over $\mathbb{Z}[1/2]$ such that $\pi_1(S)/\pi_1(S)^2$ is finite. (Here $\pi_1(S)$ denotes the fundamental group of S with respect to some base point.)*

Let $\mathcal{E}$ be an elliptic curve over S. Assume that there exists a sequence $(\pi_i)_{i \in \mathbb{N}_0}$ of minimal covers $\pi_i : \mathcal{C}_i \to \mathcal{E}$ (where the $\mathcal{C}_i$ are curves of genus 2 over S) with pairwise distinct degrees and $\deg(\pi_0) = 2$ as well as $\deg(\pi_i) \geq 5$ for $i \geq 1$ such that the branch loci of the π_i are all equal.

Let $\mathcal{C} := \mathcal{C}_0$. Then there exists a connected Galois cover $T \to S$ with Galois group a (finite) elementary abelian 2-group such that the curve $\mathcal{C}_T$ over T has a pro-Galois curve cover whose Galois group $\mathcal{G}$ fits into an exact sequence

$$1 \to \prod_{i=1}^{\infty} A_{n_i} \to \mathcal{G} \to (\mathbb{Z}/2\mathbb{Z})^r \to 1$$

for some $r \leq 4$, where $n_i := \deg(\pi_i)$ and A_{n_i} is the alternating group on n_i elements.

The rest of this section is devoted to a proof of this proposition.

We first show that any integral, regular scheme S of finite type over $\mathbb{Z}[1/2]$ the assumption that $\pi(S)/\pi(S)^2$ is finite is satisfied. Thereby, we give some background information on this condition as well.

Let S be any scheme over $\mathbb{Z}[1/2]$. Then we have an exact sequence

$$0 \to \Gamma(S, \mathcal{O}_S)^*/\Gamma(S, \mathcal{O}_S)^{*2} \to \mathrm{Hom}(\pi_1(S), \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Pic}(S)[2] \to 0\ ;$$

see [17, Proposition 4.11] together with [17, Corollary 4.7]. We thus see that $\pi_1(S)/\pi_1(S)^2$ is finite if and only if both $\mathrm{Pic}(S)[2]$ and $\Gamma(S, \mathcal{O}_S)^*/\Gamma(S, \mathcal{O}_S)^{*2}$ are finite.

If now S is an integral, normal scheme of finite type over $\mathbb{Z}$, both groups $\mathrm{Pic}(S)[2]$ and $\Gamma(S, \mathcal{O}_S)^*/\Gamma(S, \mathcal{O}_S)^{*2}$ are in fact finite, and hence so is $\pi_1(S)/\pi_1(S)^2$; see [13, Theorems 7.4 and 7.5].

Now let the assumptions of the proposition be satisfied. Let Δ be the branch locus of each of the π_i. As all residue characteristics are distinct from 2 and $\deg(\pi_0) = 2$, for any $s \in S$ $(\pi_0)_s$ is generically étale, in particular, $(\pi_0)_s$ is finite, and all branch points in codimension 1 on $\mathcal{E}$ lie in the generic fiber over S.

More precisely, we have by [1, Lemma 3.13]

Lemma 4.2: *The canonical morphism $\Delta \to S$ is an étale cover of degree 2.*

Note that the curve $\mathcal{E}$ over S gives rise to an elliptic curve over $\kappa(S)$. In particular, the field extension $\kappa(\mathcal{E})|\kappa(S)$ is regular. Let $L_i|\kappa(\mathcal{E})$ be the Galois closure of the extension of function fields $\pi_i^{\#} : \kappa(\mathcal{E}) \hookrightarrow \kappa(\mathcal{C}_i)$.

Let us fix some compositum $L_i\overline{\kappa(S)}$ of L_i and $\overline{\kappa(S)}$ over $\kappa(S)$.

Lemma 4.3: *The Galois group of $L_i\overline{\kappa(S)}|\kappa(\mathcal{E})\overline{\kappa(S)}$ is isomorphic to the symmetric group S_{n_i}. In particular, the Galois group of $L_i|\kappa(\mathcal{E})$ is isomorphic to S_{n_i}, and $L_i|\kappa(S)$ is regular.*

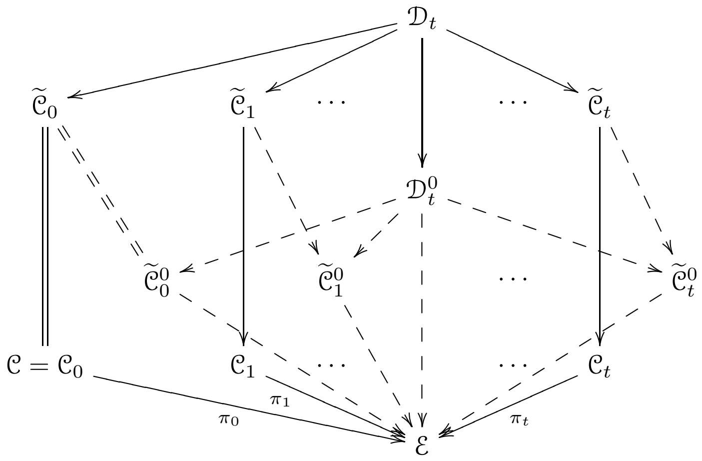
Proof. As the extension of function fields $\kappa(\mathcal{C}_i)\overline{\kappa(S)}|\kappa(\mathcal{E})\overline{\kappa(S)}$ over $\overline{\kappa(S)}$ has 2 branched places, the conorm of each of these places has the form $\mathfrak{P}_1^2\mathfrak{P}_2 \cdots \mathfrak{P}_{n_i-1}$, where the $\mathfrak{P}_j$ are pairwise distinct places. It follows that the Galois group of $L_i\overline{\kappa(S)}|\kappa(\mathcal{E})\overline{\kappa(S)}$ (seen as permutation group on n_i elements) contains a transposition.

Moreover, as $\pi_i : \mathcal{C}_i \to \mathcal{E}$ is minimal, $\kappa(\mathcal{C}_i)|\kappa(\mathcal{E})$ has no proper intermediate fields, and the same is true for $\kappa(\mathcal{C}_i)\overline{\kappa(S)}|\kappa(\mathcal{E})\overline{\kappa(S)}$.

Hence the Galois group of $L_i\overline{\kappa(S)}|\kappa(\mathcal{E})\overline{\kappa(S)}$ is a primitive transitive subgroup of S_{n_i} with a transposition, and hence it is equal to S_{n_i}, see [23, Theorem 13.3]. ∎

Let $L_i^0 := L_i^{A_{n_i}}$. Let $\widetilde{\mathcal{C}}_i$ be the normalization of $\mathcal{E}$ in L_i, and let $\widetilde{\mathcal{C}}_i^0$ be the normalization of $\mathcal{E}$ in L_i^0.

Let us fix inclusions of $\kappa(\widetilde{\mathcal{C}}_i)|\kappa(\mathcal{E})$ into some algebraic closure of $\kappa(\mathcal{E})$. For $t \in \mathbb{N}$, let $\mathcal{D}_t$ be the normalization of $\mathcal{E}$ in the compositum of $\kappa(\widetilde{\mathcal{C}}_0), \dots, \kappa(\widetilde{\mathcal{C}}_t)$ over $\kappa(\mathcal{E})$, and let $\mathcal{D}_t^0$ be the normalization of $\mathcal{E}$ in the compositum of $\kappa(\widetilde{\mathcal{C}}_0^0), \dots, \kappa(\widetilde{\mathcal{C}}_t^0)$ over $\kappa(\mathcal{E})$ (both composita with respect to the inclusions into the fixed algebraic closure of $\kappa(\mathcal{E})$). The extensions $\kappa(\mathcal{D}_t)|\kappa(\mathcal{E})$ and $\kappa(\mathcal{D}_t^0)|\kappa(\mathcal{E})$ are Galois, and $\kappa(\mathcal{D}_t^0)|\kappa(\mathcal{E})$ has as Galois group an elementary abelian 2-group.



Our goal is now to prove the following proposition.

Proposition 4.4: *For all $t \in \mathbb{N}$, we have:*

a) *The morphisms $\mathcal{D}_t \to \mathcal{C}$ and $\mathcal{D}_t^0 \to \mathcal{C}$ are Galois covers.*

b) *The Galois group of $\mathcal{D}_t^0 \to \mathcal{C}$ is an elementary abelian 2-group.*

c) *The restrictions $\mathrm{Gal}(\kappa(\mathcal{D}_t)|\kappa(\mathcal{D}_t^0)) \to \mathrm{Gal}(\kappa(\widetilde{\mathcal{C}}_i)|\kappa(\widetilde{\mathcal{C}}_i^0))$ induce an isomorphism*

$$\mathrm{Gal}(\mathcal{D}_t \to \mathcal{D}_t^0) \simeq \mathrm{Gal}(\kappa(\mathcal{D}_t)|\kappa(\mathcal{D}_t^0)) \tilde{\to} \prod_{i=1}^{t} \mathrm{Gal}(\kappa(\widetilde{\mathcal{C}}_i)|\kappa(\widetilde{\mathcal{C}}_i^0)) \approx \prod_{i=1}^{t} A_{n_i}.$$

d) *Let F_t be the algebraic closure of $\kappa(S)$ in $\kappa(\mathcal{D}_t^0)$, let S_t^0 be the normalization of S in F_t. Then $S_t^0 \to S$ is a Galois cover, and $\mathcal{D}_t^0$ as well* *as $\mathcal{D}_t$ are in a canonical way curves over S_t^0. The induced morphism $\mathcal{D}_t^0 \to \mathcal{C}_{S_t^0}$ is a Galois curve cover over S_t^0 with Galois group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$ for some $r \leq 4$. We have an exact sequence of Galois groups*

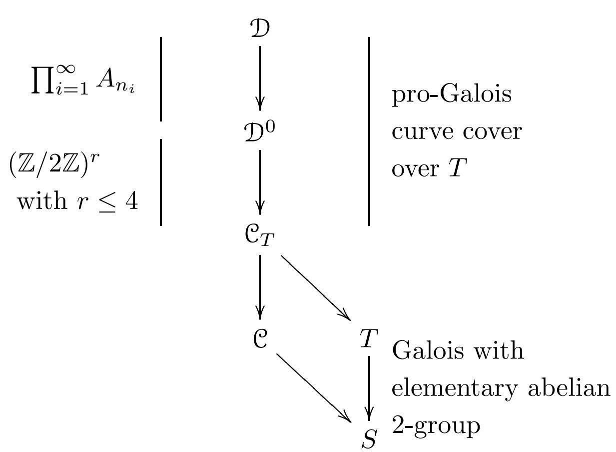
$$1 \to \mathrm{Gal}(\mathcal{D}_t^0 \to \mathcal{C}_{S_t^0}) \to \mathrm{Gal}(\mathcal{D}_t^0 \to \mathcal{C}) \to \mathrm{Gal}(S_t^0 \to S) \to 1.$$

In particular, $\mathrm{Gal}(S_t^0 \to S)$ *is an elementary abelian 2-group.*

Let us for the moment assume that we have proved Proposition 4.4. Then it is not difficult to derive Proposition 4.1:

For any $t \in \mathbb{N}$, the cover $\mathcal{D}_t^0 \to \mathcal{C}$ is a composite of the cover $\mathcal{D}_t^0 \to \mathcal{C}_{S_t^0}$ and the cover $\mathcal{C}_{S_t^0} \to \mathcal{C}$. By the last item of Proposition 4.4 and the assumption that $\pi_1(S)/\pi_1(S)^2$ is finite, there exists some $t_0 \in \mathbb{N}$ such that for $t > t_0$, the canonical morphism $\mathcal{D}_t^0 \to \mathcal{D}_{t_0}^0$ is an isomorphism.

Let $\mathcal{D}^0 := \mathcal{D}_{t_0}^0 = \varprojlim \mathcal{D}_t^0$, let $T := S_{t_0}^0$. Then for all $t \geq t_0$, $\mathcal{D}_t$ is a curve over T, and so is $\mathcal{C}_T$. Let $\mathcal{D} := \varprojlim \mathcal{D}_t$. Then $\mathcal{D}$ is a pro-Galois curve cover of $\mathcal{C}_T$ of infinite degree. Moreover, its Galois group is an extension of a group of the form $(\mathbb{Z}/2\mathbb{Z})^r$ with $r \leq 4$ by $\prod_{i=1}^{\infty} \mathrm{Gal}(\kappa(\mathcal{C}_i)|\kappa(\mathcal{C}_i^0)) \approx \prod_{i=1}^{\infty} A_{n_i}$. This implies Proposition 4.1.



Now we give the proof of Proposition 4.4. It is divided into several lemmas.

Proof of Assertion a). Let us recall Abhyankar's Lemma.

LEMMA 4.5 (Abhyankar's Lemma): *Let K be a field, $L|K, M|K$ finite separable extensions of K, $N = LM$ a compositum of M and L over K. Let v be a discrete* *valuation of* N, v_M, v_L, v_K *the restrictions of* v *to* M, L, K *respectively. Assume that the extensions* $v_M|v_K$ *and* $v_L|v_K$ *are tame and that* $e(v_M|v_K)|e(v_L|v_K)$. *Then* $v|v_L$ *is unramified.*

For a proof see [22, Proposition III.8.9] (the assumptions in [22, Proposition III.8.9] that $M|K$ be an extension of function fields in one variable and v a valuation of function fields is not necessary).

Lemma 4.6: *The morphisms* $\widetilde{\mathcal{C}}_i \to \mathcal{E}$ *and* $\mathcal{D}_t \to \mathcal{E}$ *are finite, the branched points in codimension 1 are the generic points of* Δ, *and the corresponding ramification indices are 2.*

Proof. The first statement follows from the following general fact: if one normalizes an integral noetherian scheme in a finite separable extension of its function field, the canonical morphism is finite (cf., [15, Chapter 4, Proposition 1.25]). By Abhyankar's Lemma, the ramified points in codimension 1 are the generic points of Δ and the ramification indices divide 2. As the corresponding extensions of function fields are Galois, the ramification indices for points above the branched points in codimension 1 are 2. ∎

Lemma 4.7: *The morphisms* $\mathcal{D}_t \to \widetilde{\mathcal{C}}_0 = \mathcal{C}$ *are étale covers.*

Proof. Let $t \in \mathbb{N}$ be fixed. By Lemma 4.6, the branch loci of $\mathcal{D}_t \to \mathcal{E}$ and $\mathcal{C} \to \mathcal{E}$ are equal, and the ramification indices of the branched points in codimension 1 are dividing 2. As $L_0|\kappa(\mathcal{E})$ has degree 2, this implies that for all points x in codimension 1 of $\mathcal{E}$, all ramification indices of $\mathcal{D}_t \to \mathcal{E}$ at x divide all ramification indices of $\widetilde{\mathcal{C}}_0 \to \mathcal{E}$ at x.

By Abhyankar's Lemma, $\mathcal{D}_t \to \mathcal{C}$ is unramified at all points of codimension 1. As S is regular and $\mathcal{C}$ is smooth over S, $\mathcal{C}$ is regular (in particular normal).

By "purity of the branch locus" ([8, Exposé X, 3.1]), $\mathcal{D}_t \to \mathcal{C}$ is unramified everywhere, and because $\mathcal{C}$ is normal, it is étale; see [8, Exposé I, Corollaire 9.11]. ∎

Lemma 4.8: *The morphisms* $\mathcal{D}_t \to \mathcal{C}$, $\mathcal{D}_t \to \mathcal{D}_t^0$ *and* $\mathcal{D}_t^0 \to \mathcal{C}$ *are Galois covers.*

Proof. By the previous lemma, we already know that the morphisms $\mathcal{D}_t \to \mathcal{C}$ are étale covers. Moreover, the corresponding extensions of functions fields are Galois. As $\mathcal{C}$ (respectively $\mathcal{D}_t^0$) is normal, this implies that the covers $\mathcal{D}_t \to \mathcal{C}$ and $\mathcal{D}_t \to \mathcal{D}_t^0$ are Galois. This, together with [8, Corollaire 3.4] implies that

$\mathcal{D}_t^0 \to \mathcal{C}$ is étale. It is Galois because the corresponding extension of function fields is Galois (and $\mathcal{C}$ is normal). ∎

Proof of Assertions b) and c). We need a group theoretical lemma.

Lemma 4.9: *Let $G_1, \ldots, G_t$ be groups such that for any i, j with $i \neq j$, there does not exist any non-trivial simple group which is a quotient group of both G_i and G_j. Let G be a group with surjective homomorphisms $p_i : G \to G_i$. Then the induced homomorphism $p : G \to \prod_{i=1}^t G_i$ is surjective.*

Proof. The proof can be done by induction on t. For $t = 2$, the proof is as follows.

Let $N := \langle \ker(p_1) \cup \ker(p_2) \rangle$ — this is a normal subgroup of G. We have canonical surjective homomorphisms $G_i \simeq G/\ker(p_i) \to G/N$. By assumption, G/N is trivial, that is, $G = N$.

This implies that $p_1|_{\ker(p_2)} : \ker(p_2) \to G_1$ and $p_2|_{\ker(p_1)} : \ker(p_1) \to G_2$ are surjective. This in turn implies that the image of p contains $G_1 \times \{1\}$ and $\{1\} \times G_2$, thus p is surjective. ∎

Now we can proceed with the proof of Proposition 4.4.

By construction, $\kappa(\mathcal{D}_t^0)|\kappa(\mathcal{E})$ is Galois with Galois group an elementary abelian 2-group. It follows that the Galois group of $\mathcal{D}_t^0 \to \mathcal{C}$ is an elementary abelian 2-group. This proves Assertion b).

The covers $\mathcal{D}_t \to \mathcal{D}_t^0$ are more interesting.

Lemma 4.10: *The restrictions $\mathrm{Gal}(\kappa(\mathcal{D}_t)|\kappa(\mathcal{D}_t^0))) \to \mathrm{Gal}(\kappa(\widetilde{\mathcal{C}}_i)|\kappa(\widetilde{\mathcal{C}}_i^0))$ are surjective and induce an isomorphism*

$$\mathrm{Gal}(\mathcal{D}_t \to \mathcal{D}_t^0) \simeq \mathrm{Gal}(\kappa(\mathcal{D}_t)|\kappa(\mathcal{D}_t^0)) \tilde{\to} \prod_{i=1}^t \mathrm{Gal}(\kappa(\widetilde{\mathcal{C}}_i)|\kappa(\widetilde{\mathcal{C}}_i^0)) \approx \prod_{i=1}^t A_{n_i}.$$

Proof. The group $\mathrm{Gal}(\kappa(\widetilde{\mathcal{C}}_0)|\kappa(\widetilde{\mathcal{C}}_0^0))$ is trivial, so let $i \geq 1$. The group $\mathrm{Gal}(\kappa(\widetilde{\mathcal{C}}_i)|\kappa(\widetilde{\mathcal{C}}_i^0))$ is isomorphic to A_{n_i}, and the group $\mathrm{Gal}(\kappa(\mathcal{D}_i^0)|\kappa(\widetilde{\mathcal{C}}_i^0))$ is an elementary abelian 2-group. As $n_i \geq 5$ (because $i \geq 1$), the group A_{n_i} is simple (see [21, Theorem 3.15]) and in particular has no non-trivial elementary abelian 2-group as quotient. By Lemma 4.9, the extensions $\kappa(\mathcal{D}_t^0)$ and $\kappa(\widetilde{\mathcal{C}}_i)$ are linearly disjoint over $\kappa(\widetilde{\mathcal{C}}_i^0)$ (inside $\kappa(\mathcal{D}_t)$) (this will also be used in the proof of Lemma 4.12). In particular, the restriction map $\mathrm{Gal}(\kappa(\mathcal{D}_t^0)\,\kappa(\widetilde{\mathcal{C}}_i)|\kappa(\mathcal{D}_t^0)) \to \mathrm{Gal}(\kappa(\widetilde{\mathcal{C}}_i)|\kappa(\widetilde{\mathcal{C}}_i^0))$ is an isomorphism.

Again by Lemma 4.9, the induced homomorphism $\mathrm{Gal}(\kappa(\mathcal{D}_t)|\kappa(\mathcal{D}_t^0)) \to \prod_{i=1}^t \mathrm{Gal}(\kappa(\widetilde{\mathcal{C}}_i)|\kappa(\widetilde{\mathcal{C}}_i^0)) \approx \prod_{i=1}^t A_{n_i}$ is surjective. It is obvious that it is injective. ∎

Proof of Assertion d). Again we first need a group theoretical lemma.

LEMMA 4.11: *Let $G_1, \ldots, G_t$ be finite groups such that any i, j with $i \neq j$, there is no simple group which occurs as a composition factor of both G_i and G_j. Let G be a group with a homomorphism $\varphi : \prod_{i=1}^t G_i \to G$ such that for $i = 1, \ldots, t$, the restriction of φ to G_i (regarded as a subgroup of $\prod_{i=1}^t G_i$) has kernel N_i. Then φ has kernel $\prod_{i=1}^t N_i$ (regarded as a subgroup of $\prod_{i=1}^t G_i$).*

Proof. Recall that if G is a finite group and $N \lhd G$ is a normal subgroup, then composition series of both N and G/N in a canonical way give rise to a composition series of G. In particular, the set of composition factors of G is the union of the sets of composition factors of N and G/N.

The assumption and this remark imply that for any $j = 2, \ldots, t$, there is no simple group which occurs in the composition series of both $\prod_{i=i}^{j-1} G_i$ and G_j. Because of this, the general case follows by induction from the case for $t = 2$.

In this case, the proof is as follows.

Obviously, $N_1 \times N_2$ in contained in the kernel of φ. The group $\varphi(G_1 \times \{1\}) \cap \varphi(\{1\} \times G_2)$ is normal in both $\varphi(G_1 \times \{1\})$ and $\varphi(\{1\} \times G_2)$.

However by assumption and the remark at the beginning of the proof, the groups $\varphi(G_1 \times \{1\})$ and $\varphi(\{1\} \times G_2)$ cannot contain a non-trivial common normal subgroup. This implies that the group $\varphi(G_1 \times \{1\}) \cap \varphi(\{1\} \times G_2)$ is trivial.

Now if $\varphi(g_1, g_2) = 1$, then $\varphi(g_1, 1) = \varphi(1, g_2^{-1}) \in \varphi(G_1 \times \{1\}) \cap \varphi(\{1\} \times G_2)$. Thus this has to be 1. By the definition of N_1 and N_2, $g_1 \in N_1$, $g_2 \in N_2$, that is, $(g_1, g_2) \in N_1 \times N_2$. ∎

LEMMA 4.12: *Let F_t be the algebraic closure of $\kappa(S)$ in $\kappa(\mathcal{D}_t^0)$. Then $\kappa(\mathcal{D}_t)|F_t$ is regular.*

Proof. As we have seen in the proof of Lemma 4.10, the fields $\kappa(\mathcal{D}_t^0)$ and $\kappa(\widetilde{\mathcal{C}}_i)$ are linearly disjoint extensions of $\kappa(\widetilde{\mathcal{C}}_i^0)$. Let $F_{t,i}$ be the algebraic closure of F_t in $\kappa(\mathcal{D}_t^0)\kappa(\widetilde{\mathcal{C}}_i)$. If we apply Lemma 4.11 to the restriction map $\mathrm{Gal}(\kappa(\mathcal{D}_t^0)\kappa(\widetilde{\mathcal{C}}_i)|\kappa(\widetilde{\mathcal{C}}_i^0)) \to \mathrm{Gal}(F_{t,i}\kappa(\widetilde{\mathcal{C}}_i^0)|\kappa(\widetilde{\mathcal{C}}_i^0))$, we obtain that

$F_{t,i}\kappa(\tilde{\mathcal{C}}_i^0) = F_t\kappa(\tilde{\mathcal{C}}_i^0)$. As $\kappa(\tilde{\mathcal{C}}_i^0)|\kappa(S)$ is regular, we obtain $F_{t,i} = F_t$, that is, $\kappa(\mathcal{D}_t^0)\kappa(\widetilde{\mathcal{C}}_i)|F_t$ is regular.

By the structure of $\mathrm{Gal}(\kappa(\mathcal{D}_t)|\kappa(\mathcal{D}_t^0))$ and again Lemma 4.11, this implies that $\kappa(\mathcal{D}_t)|F_t$ is regular. ∎

Lemma 4.13: *Let $q_t : \mathcal{D}_t^0 \to S$ be the structure morphism, and let S_t^0 be the normalization of S in F_t. Then S_t^0 is equal to $\mathbf{Spec}(q_{t*}(\mathcal{O}_{\mathcal{D}_t^0}))$, and the canonical morphism $S_t^0 \to S$ is an étale cover.*

Proof. As $\mathcal{D}_t^0$ is smooth and proper over S, for all $s \in S$, the ring of global sections of the fiber $(\mathcal{D}_t^0)_s$ is a finite separable algebra over $\kappa(s)$. This together with [9, III (7.8.7)] implies that $\mathbf{Spec}(q_{t*}(\mathcal{O}_{\mathcal{D}_t^0})) \to S$ is an étale cover. Again, by [9, III (7.8.7)] applied to the generic point of S, one sees that the total ring of fractions of $\mathbf{Spec}(q_{t*}(\mathcal{O}_{\mathcal{D}_t^0}))$ is F_t. This implies that $\mathbf{Spec}(q_{t*}(\mathcal{O}_{\mathcal{D}_t^0}))$ is the normalization of S in F_t, that is, it is S_t^0; see [8, Corollaire 10.2]. ∎

We can consider $\mathcal{D}_t^0$ as an S_t^0-scheme (Stein factorization); let $r_t : \mathcal{D}_t^0 \to S_t^0$ be the structure morphism. Then $\mathcal{O}_{S_t^0} = r_{t*}(\mathcal{O}_{\mathcal{D}_t^0})$, and $\mathcal{D}_t^0$ has connected and non-empty geometric fibers over S_t^0 (see [9, III (4.3.1), (4.3.4)]).

By the universal property of the product, $\mathcal{D}_t^0 \to \mathcal{C}$ factors through $\mathcal{C}_{S_t^0} \to \mathcal{C}$.

Lemma 4.14: *$\mathcal{D}_t^0$ is a curve over S_t^0 and $\mathcal{D}_t^0 \to \mathcal{C}_{S_t^0}$ is an étale curve cover.*

Proof. The morphism $\mathcal{D}_t^0 \to \mathcal{C}_{S_t^0}$ is an étale cover, because $\mathcal{D}_t^0 \to \mathcal{C}$ and $\mathcal{C}_{S_t^0} \to \mathcal{C}$ are étale covers. This implies that $\mathcal{D}_t^0$ is smooth and proper over S_0^t. It is a curve because it has connected and 1-dimensional geometric fibers. ∎

As $S_t^0 \to S$ is an étale cover and $F_t|\kappa(S)$ is Galois with Galois group a quotient of that of $\kappa(\mathcal{C}_t^0)|\kappa(\mathcal{C})$ (as one easily sees) and S is normal, $S_t^0 \to S$ is a Galois cover with Galois group an elementary abelian 2-group. The Galois group is canonically isomorphic to that of $\mathcal{C}_{S_t^0} \to \mathcal{C}$ because base-change does not change the Galois group.

By Galois theory the cover $\mathcal{D}_t^0 \to \mathcal{C}_{S_t^0}$ is Galois, and we have the exact sequence

$$1 \to \mathrm{Gal}(\mathcal{D}_t^0 \to \mathcal{C}_{S_t^0}) \to \mathrm{Gal}(\mathcal{D}_t^0 \to \mathcal{C}) \to \mathrm{Gal}(\mathcal{C}_{S_t^0} \to \mathcal{C}) \to 1$$

with $\mathrm{Gal}(S_t^0 \to S) \simeq \mathrm{Gal}(\mathcal{C}_{S_t^0} \to \mathcal{C})$, where all groups are elementary abelian 2-groups. By pull-back to a geometric fiber, we see that $\mathrm{Gal}(\mathcal{D}_t^0 \to \mathcal{C}_{S_t^0})$ is a group of the form $(\mathbb{Z}/2\mathbb{Z})^r$ with $r \leq 2 \cdot g_{\mathcal{C}} = 4$.

Now let S_t be analogously defined to S_t^0 for $\mathcal{D}_t$. Then $\mathcal{D}_t$ has connected geometric fibers over S_t and S_t is étale over S_t^0. By Lemma 4.12 and [9, III (7.8.7)], the generic fiber of $\mathcal{D}_t$ over S_t^0 is geometrically connected. On the other hand, again by [9, III (7.8.7)] the number of connected components in each geometric fiber of $\mathcal{D}_t$ over S_t^0 is equal to the degree of S_t over S_t^0. This implies that $S_t \tilde{\rightarrow} S_t^0$. The fact that $\mathcal{D}_t \to \mathcal{D}_t^0$ is an étale cover (Lemma 4.8) implies that $\mathcal{D}_t$ is a curve over S_t^0. This completes the proof of Proposition 4.4.

5. Complementary Results

5.1. Abelian Surfaces Isogenous and Isomorphic to a Product of Elliptic Curves. Recall that if $\mathcal{C} \to \mathcal{E}$ is a minimal cover of degree n over some scheme S, the Jacobian of $\mathcal{C}$ is as principally polarized abelian variety isomorphic to $((\mathcal{E} \times \mathcal{E}')/\mathrm{Graph}(-\psi), \lambda)$ where $\mathcal{E}'$ is an elliptic curve over S, $\psi : \mathcal{E}[n] \to \mathcal{E}'[n]$ is an isomorphism which is anti-isometric with respect to the Weil pairing and λ is the (principal) polarization whose pull-back to $\mathcal{E} \times \mathcal{E}'$ is n times the product polarization; see [12].

An important special case is that ψ is induced by an isogeny $\tau : \mathcal{E} \to \mathcal{E}'$ (necessarily of degree coprime to n). In this case $(\mathcal{E} \times \mathcal{E}')/\mathrm{Graph}(-\psi)$ is in fact always isomorphic to $\mathcal{E} \times \mathcal{E}'$ as can be seen by the following easy generalization of the proof of Proposition 3.5.

Let n is some natural number, let $\mathcal{E}$, $\mathcal{E}'$ two elliptic curves over a scheme S, let $\tau : \mathcal{E} \to \mathcal{E}'$ be any isogeny of degree N coprime to n. (In Proposition 3.5 we treated the case that n is 2 and N is odd.) As in Section 3, let $\psi := \tau|_{\mathcal{E}[n]}$.

Then the isogeny $\Phi : \mathcal{E} \times \mathcal{E}' \to \mathcal{E} \times \mathcal{E}'$ given by the matrix $\left(\begin{smallmatrix} n & 0 \\ \tau & 1 \end{smallmatrix}\right)$ has kernel $\mathrm{Graph}(-\psi)$, thus it induces in isomorphism

$$(\mathcal{E} \times \mathcal{E}')/\,\mathrm{Graph}(-\psi) \tilde{\rightarrow} \mathcal{E} \times \mathcal{E}'. \tag{2}$$

Assume now that ψ is anti-isometric with respect to the Weil pairing, i.e., that N is congruent to -1 modulo n. Then $(\mathcal{E} \times \mathcal{E}')/\mathrm{Graph}(-\psi)$ has a principal polarization λ whose pull-back to $\mathcal{E} \times \mathcal{E}'$ is $n\ \mathrm{id}_{\mathcal{E} \times \mathcal{E}'}$; see [12, Proposition 5.7]. (As in Section 3, we identify elliptic curves (and products of elliptic curves) with their duals, so that the canonical (product) polarizations become the identity.)

Under isomorphism (2), the polarization λ corresponds to the principal polarization $\widetilde{\lambda}$ on $\mathcal{E} \times \mathcal{E}'$ whose pull-back with Φ is also $n\ \mathrm{id}_{\mathcal{E} \times \mathcal{E}'}$, i.e., we have $\widehat{\Phi}\,\widetilde{\lambda}\,\Phi = n\ \mathrm{id}_{\mathcal{E} \times \mathcal{E}'}$. It follows that $\widetilde{\lambda} = n\,\widehat{\Phi}^{-1}\,\Phi^{-1}$, and consequently $\widetilde{\lambda}$ is given by the matrix

$$\frac{1}{n}\begin{pmatrix} 1 & -\hat{\tau} \\ 0 & n \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\tau & n \end{pmatrix} = \begin{pmatrix} \frac{1+N}{n} & -\hat{\tau} \\ -\tau & n \end{pmatrix}. \tag{3}$$

In particular any principally polarized abelian surface considered in [4, Proposition 3.1] is isomorphic to a product of elliptic curves with a polarization given by (3).

5.2. The Number Field Case. It is interesting to note that the assumptions of Proposition 4.1 cannot be satisfied for S equal to on open part of $\mathrm{Spec}(\mathcal{O}_K[1/2])$, where $\mathcal{O}_K$ is the principal order in a number field K. Indeed, this is contradicted by the following proposition.

Proposition 5.1: *Let K be a number field, let S an open part of $\mathrm{Spec}(\mathcal{O}_K[1/2])$, and let $\mathcal{E}$ be an elliptic curve over S. Then there does not exist a sequence $(\pi_i)_{i\in\mathbb{N}_0}$ of minimal covers $\pi_i : \mathcal{C}_i \to \mathcal{E}$ (where the $\mathcal{C}_i$ are curves of genus 2) with pairwise distinct degrees and the same branch locus which is étale of degree 2 over S.*

Proof. By Faltings' proof of the Shafarevich Conjecture ([2]), there exist only finitely many isomorphism classes of curves of genus 2 over S. It thus suffices to prove the proposition under the assumption that the $\mathcal{C}_i$ are equal to each other. Now the conclusion is implied by the following proposition. ∎

Proposition 5.2: *Let K be a number field, E an elliptic curve and C a curve of genus 2 over K. Then there does not exist a sequence $(\pi_i)_{i\in\mathbb{N}_0}$ of minimal covers $\pi_i : C \to E$ with pairwise distinct degrees and the same branch locus which has degree 2 over K.*

Proof. Assume there exists a sequence $(\pi_i)_{i\in\mathbb{N}_0}$ of minimal covers with pairwise distinct degrees. Let Δ be the common branch locus, let V_i the ramification locus of π_i. By the assumption and the Hurwitz genus formula both Δ and V_i have degree 2, thus the canonical maps $V_i \to \Delta$ are isomorphisms. As in Proposition 2.2 the V_i are pairwise disjoint, C has infinitely many Δ-valued points. This contradicts that by Faltings' proof of the Mordell Conjecture ([2]), any curve of genus ≥ 2 over a number field has only finitely many rational points. ∎

One might ask whether there exists an elliptic curve E over a number field K and a sequence $(\pi_i)_{i\in\mathbb{N}}$ of minimal covers $\pi_i : C_i \to E$ (where the C_i are curves of genus 2 over K) with pairwise distinct degrees and equal branch loci. With similar (but easier) arguments as in Section 4, the existence of such a sequence would lead to a curve of genus 2 over the maximal elementary abelian 2-extension of K which has a pro-Galois curve cover of infinite degree.

There is the following conjecture closely related to the height conjecture for elliptic curves and so to the ABC conjecture.

Conjecture. Let E be an elliptic curve over a number field K. Then there is a number $n_0(K, E)$ such that for all elliptic curves E' over K with $E[n]$ isomorphic to $E'[n]$ for some $n > n_0(K, E)$ it follows that E and E' are isogenous (over K).

This conjecture is equivalent to Conjecture 5 in [5] in the special case that the base field is a number field. (We use that the Faltings height over a number field is effective and that given two elliptic curves E and E' over a number field such that for infinitely many natural numbers n, $E[n]$ is isomorphic to $E'[n]$, E and E' are isogenous.)

Proposition 5.3: *Let K be a number field, E an elliptic curve over K. Then* under the assumption of the above conjecture, *there are only finitely many isomorphism classes of curves of genus 2 occurring as minimal covers of E with covering degree $> n_0(K, E)$.*

Proof. Let $C \to E$ be a genus 2 cover with covering degree $n > n_0(K, E)$, $J_C \simeq (E \times E')/\operatorname{Graph}(-\psi)$. Then we have the isomorphism $\psi : E[n] \simeq E'[n]$, thus by the conjecture E is isogenous to E'. This means that J_C is isogenous to $E \times E$. By Faltings' results, there are only finitely many abelian surfaces isogenous to $E \times E$, and any of these finitely many abelian surfaces has, up to isomorphism, only finitely many principal polarizations. The result now follows by Torelli's Theorem. ∎

Together with Proposition 5.2, this proposition implies

Proposition 5.4: *Let K be a number field, E an elliptic curve over K. Then* under the assumption of the above conjecture, *there does not exist a sequence $(\pi_i)_{i\in\mathbb{N}}$ of minimal covers $\pi_i : C_i \to E$ (where the C_i are curves of genus 2) with pairwise distinct degrees and the same branch locus which has degree 2 over K.*

References

[1] C. Diem, *Families of elliptic curves with genus 2 covers of degree 2*, Collectanea Mathematica **57** (2006), 1–25.

[2] G. Faltings, *Endlichkeitssätze für abelsche Varietten über Zahlkrpern*, Inventiones Mathematicae **73** (1983), 349–366.

[3] G. Faltings and C.-L. Chai, *Degeneration of Abelian Varieties*, Springer-Verlag, Berlin, 1980.

[4] G. Frey, *On elliptic curves with isomorphic torsion structures and corresponding curves of genus* 2, in *Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993)*, Internat. Press, Cambridge, MA, 1995, pp. 79–98.

[5] G. Frey, *On Ternary Equations of Fermat Type and Relations with Elliptic Curves*, in *Modular Forms and Fermat's Last Theorem Cornell, (G. Cornell, J. Silverman and G. Stevens eds.)*, Springer-Verlag, 1997, pp. 527–548.

[6] G. Frey and E. Kani, *Projective p-adic representations of the k-rational geometric fundamental group*, Archiv der Mathematik **77** (2001), 32–46.

[7] G. Frey, E. Kani and H. Völklein, *Curves with infinite K-rational geometric fundamental group*, in *Aspects of Galois theory (Gainesville, FL, 1996)*, London Mathematical Society Lecture Note Ser., vol. 256, Cambridge Univ. Press, Cambridge, 1999, pp. 85–118.

[8] A. Grothendieck, *Revêtements étales et groupe fondamental (SGA I)*, Séminaire de Géométrie Algébrique, vol. 1960/61, Institut des Hautes Études Scientifiques, Paris.

[9] A. Grothendieck with J. Dieudonné, *Eléments de Géométrie Algébrique (I-IV)*, ch. I: Springer-Verlag, Berlin, 1971; ch. II-IV: Publications Mathématiques. Institut des Hautes Études Scientifiques 8, 11, 17, 20, 24, 28, 32, 1961–68.

[10] R. Hartshorne, *Algebraic geometry*, Springer, New York, 1977.

[11] Y. Ihara, *On unramified extensions of function fields over finite fields*, in *Galois Groups and their Representations*, Advanced Studies in Pure Mathematics, vol. 2, 1983, pp. 89–97.

[12] E. Kani, *Hurwitz spaces of genus 2 covers of an elliptic curve*, Collectanea Mathematica **54** (2003), 1–51.

[13] S. Lang, *Fundamentals of Diophantine Geometry*, Springer-Verlag, New York, 1983.

[14] S. Lang, *Algebra (Third Edition)*, Addison-Wesley, 1993.

[15] Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford University Press, Oxford, 2001.

[16] L. Merel, *Bornes pour la torsion des courbes elliptiques sur les corps de nombres*, Inventiones Mathematicae **124** (1996), 437–449.

[17] J. Milne, *Etale cohomology*, Princeton University Press, Princeton, 1980.

[18] J. Milne, *Jacobian varieties*, in *Arithmetic Geometry (G. Cornell and J. Silverman eds.)*, Springer-Verlag, New York, 1986, pp. 167–212.

[19] D. Mumford, *Geometric Invariant Theory*, Springer-Verlag, Berlin, 1965.

[20] D. Mumford, *Abelian Varieties*, Tata Institute for Fundamental Research, Oxford University Press, London, 1970.

[21] J. Rotman, *An Introduction to the Theory of Groups*, Wm. C. Brown, 1994.

[22] H. Stichtenoth, *Algebraic Function Fields and Codes*, Springer-Verlag, Berlin, 1993.

[23] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York, 1964.